Relaxation, Ergodicity and the one-dimensional sheet model

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Overview

1. Introduction — What's the problem?
2. Overview of the work by Tsuchiya et al.
3. Goal of the present work
4. Numerical method
5. Result
6. Interpretation
7. Future work

Introduction

The fundamental problem of the statistical mechanics:

- **Why one can rely on the ensemble average (the Ergod problem)?**

Ergodicity (roughly speaking): that the macroscopic quantity of a system in thermal equilibrium is obtained by taking average over the all possible phase space.

Only a handful of systems are proven to be Ergodic.

Why systems with large degree of freedoms behave thermodynamically?

Usual hand-waving

Pattern (a):
Systems with large degree of freedoms are strongly chaotic, and thus expected to be ergodic.

This is quite wrong, at least for gravitational N-body systems. We know

(a) The strength of chaoticity (Lyapunov coefficients) has nothing to do with the relaxation timescale.

(b) Systems with infinite degree of freedom would have infinitely long relaxation time.

Though the chaos is the required condition, it does not necessarily guarantee the Ergodicity.

Usual hand-waving (continued)

Pattern (b):
Grandcanonical (or quantum-mechanical) systems are stochastic, and therefore the statistical treatment is justified.

Well, then, classical isolated systems cannot be described by statistical mechanics?

This sounds quite unlikely. Isn't the evolution of globular clusters driven by thermal relaxation?

The theoretical basis of the statistical mechanics

- We know that the system does not have enough time to be Ergodic
- We know that system is deterministic, albeit chaotic

How such systems would actually behave?

- Theoretical study
- Numerical study of various systems

The FPU problem

- Connected spring and body
- Weak nonlinearity (purely linear system would not relax)

(Fermi, Pasta and Ulam 1952?) One of the first theoretical physics calculation performed on MANIAC.

They expected that the energy will quickly diffuse to different modes through mode-mode coupling.
They found the actual behavior is by far complex
— induction time
— recurrence

Interpretation of FPU problem

- mode-mode coupling is actually rather weak
- energies of modes oscillate, do not easily relax

Rather similar to what is known for resonant interaction between planets.

In other words, there is not really anything strange in apparently complex behaviors such as induction and reccurence.

FPU system is only weakly nonlinear and interaction is local

Long-range interaction and nonlinearity

Complex (or, rather regular) behavior of FPU system is due to the its character (short-range interaction and weak nonlinearity).

How systems with long range interaction and strong nonlinearity behave?

one example : gravitational N-body systems.

Gravitational N-body systems

Three-dimensional system is difficult to study

- Potential diverges at $r \longrightarrow 0$.
- Phase space is not compact.
- does not have the thermal equilibrium.

One-dimensional system looks simpler

- Potential goes to zero at $r \longrightarrow 0$.
- Phase space is compact.
- has the thermal equilibrium.

One-dimensional system

The sheet model

$$\Phi_{ij} = Gm_i m_j |x_i - x_j|$$

Force: constant (does not depend on distance)
Hamiltonian:

$$H = \frac{m}{2} \sum_{i=1}^{N} v_i^2 + 2\pi G m^2 \sum_{i<j} |x_i - x_j|, \qquad (1)$$

Simulation method:

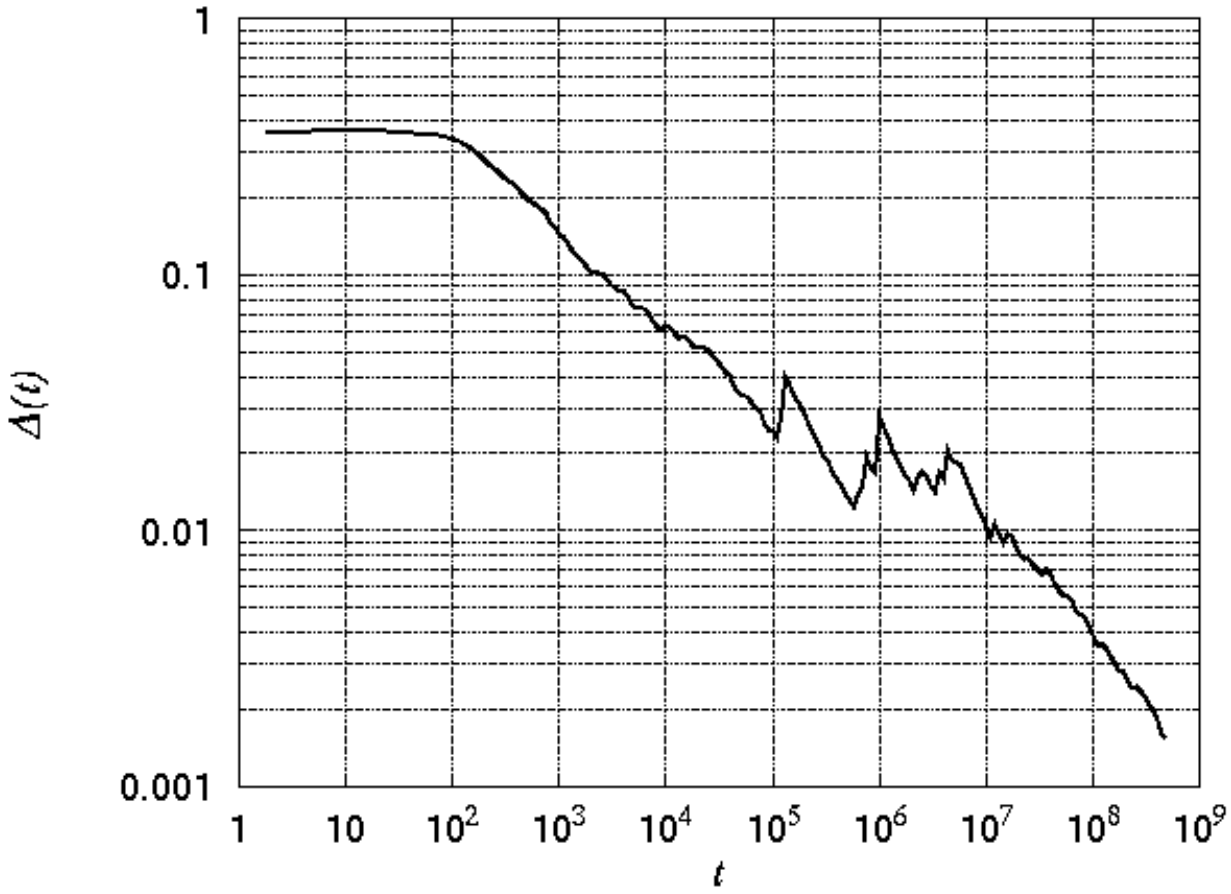
The system is solvable until two sheets collide.

Maintain the collision time for all pairs. Update the time for collided pair and their neighbours. Repeat this procedure.

Historical overview

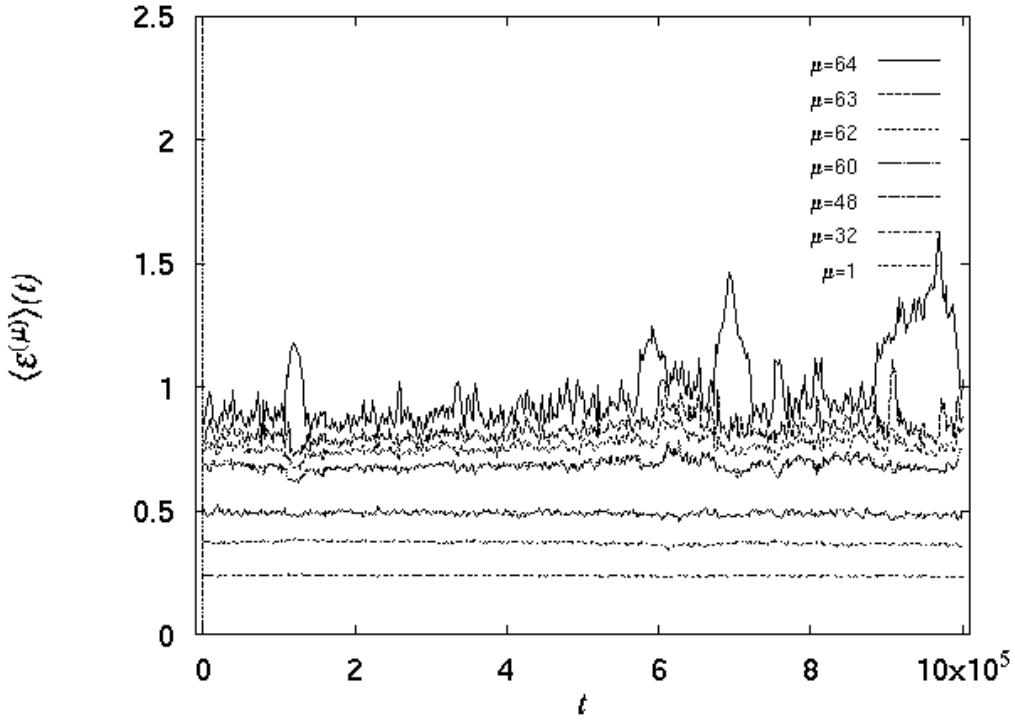
- First introduced as a simplified model for the galactic disk.
- In 1960-1980s, studied as an idealized model for the violent relaxation.
- In early 80s, it was "found" that relaxation time is $O(N)$. Previously, it had been believed to be $O(N^2)$.
- In a series of papers, Tsuchiya and his collaborators claimed that the system needs very long time to really relax (as long as $10^6 N$).

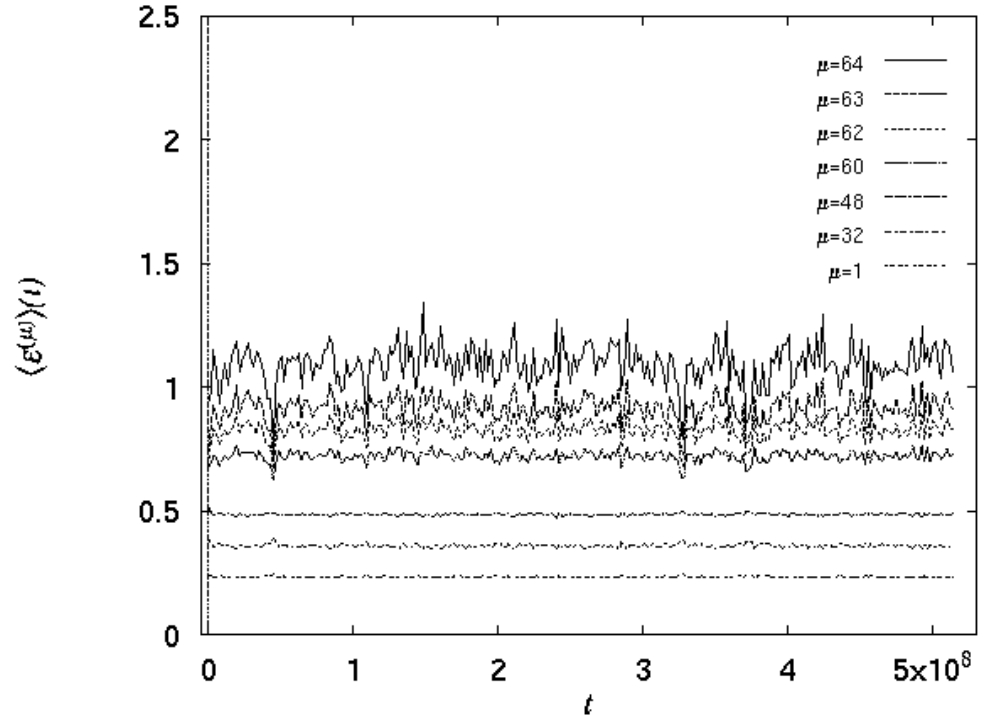
Chaotic itinerancy?



"System does not approach equilibrium monotonically"
Measured quantity:
r.m.s. of $< e_i > - e_0$, where e_i is the energy of sheet i, $e_0 = 5E/3$ is the mean energy. Averaging is taken from time 0 to t.

Behavior of the energy of the sheets





An Interpretation

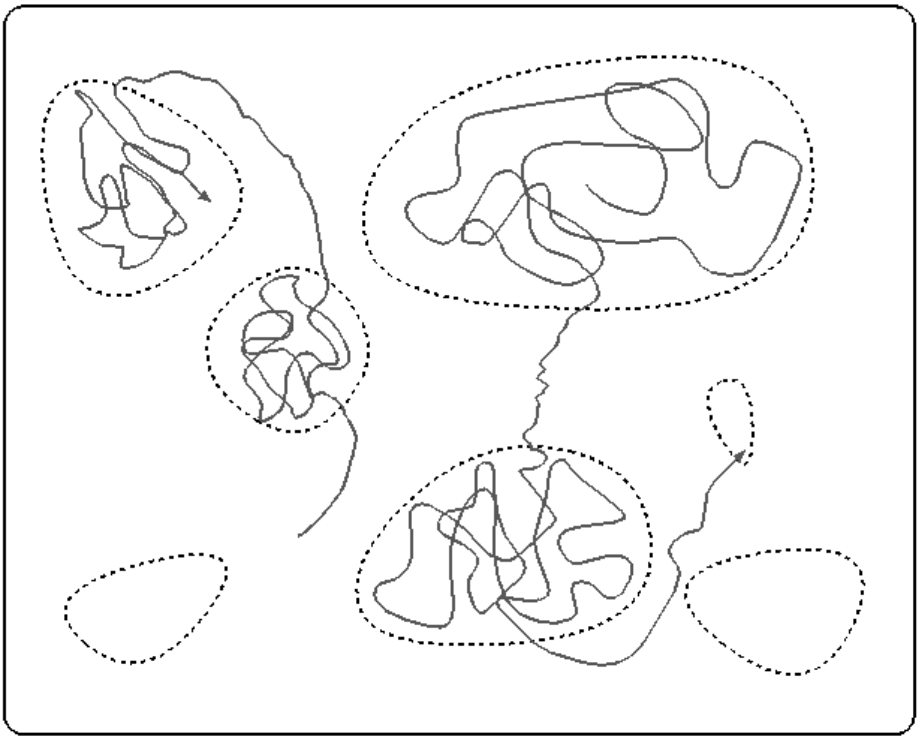


FIG. 17. Basic idea of the structure of orbits in the phase space. The solid lines indicates the orbits, which describes the states of the system. The regions which enclosed by dashed curves corresponds to the quasiequilibria. The paths which connect the quasiequilibrium regions corresponds to the transient states. Actually these structures are fractal.

Is it really that complicated?

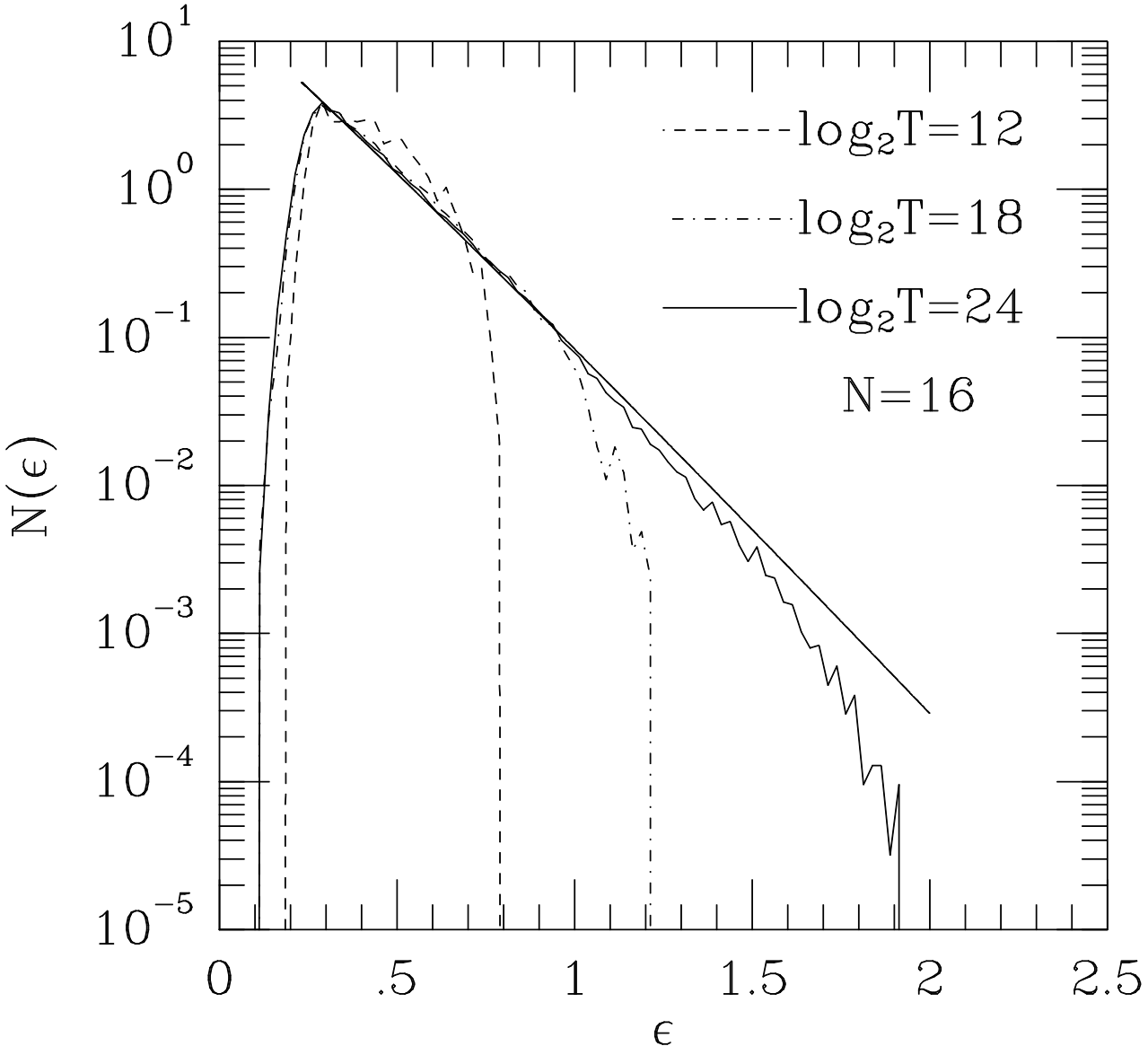
Numerical Experiment

—Follow the evolution for very long time (up to 10^7 crossing times)

—calculate thermodynamical quantities such as time-averaged distribution function and relaxation timescale.

Distribution Function

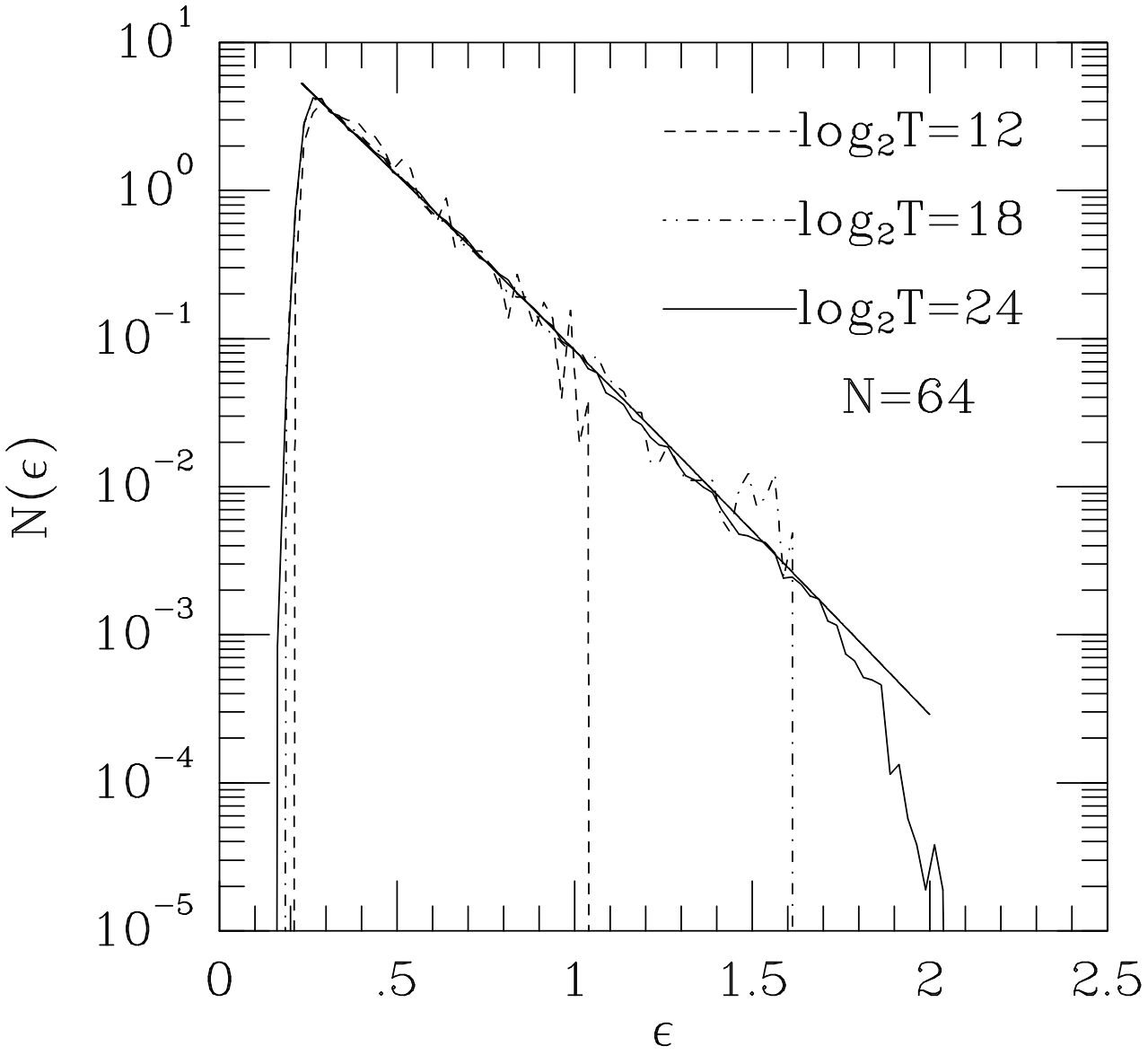
$N = 16$



Low-energy region relaxes quickly.
High-energy region is only slowly populated.

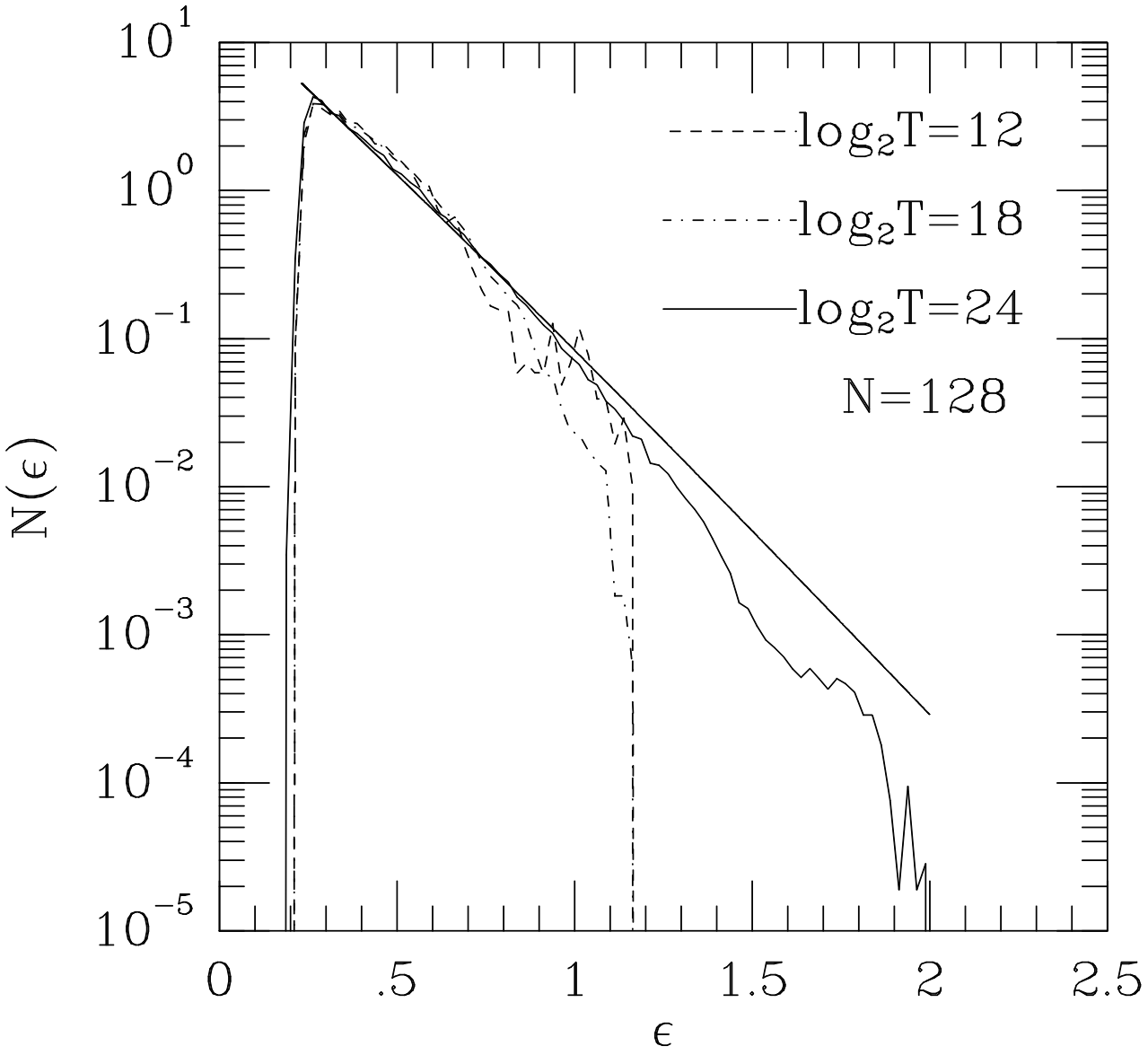
Distribution Function (2)

$$N = 64$$



Distribution Function (3)

$N = 128$



It does take very long time (much longer than N) to populate the high-energy tail of the distribution function.
WHY?

Diffusion coefficient

Local diffusion coefficients determines the local relaxation timescale. The first- and second-order diffusion coefficients are (roughly) expressed as

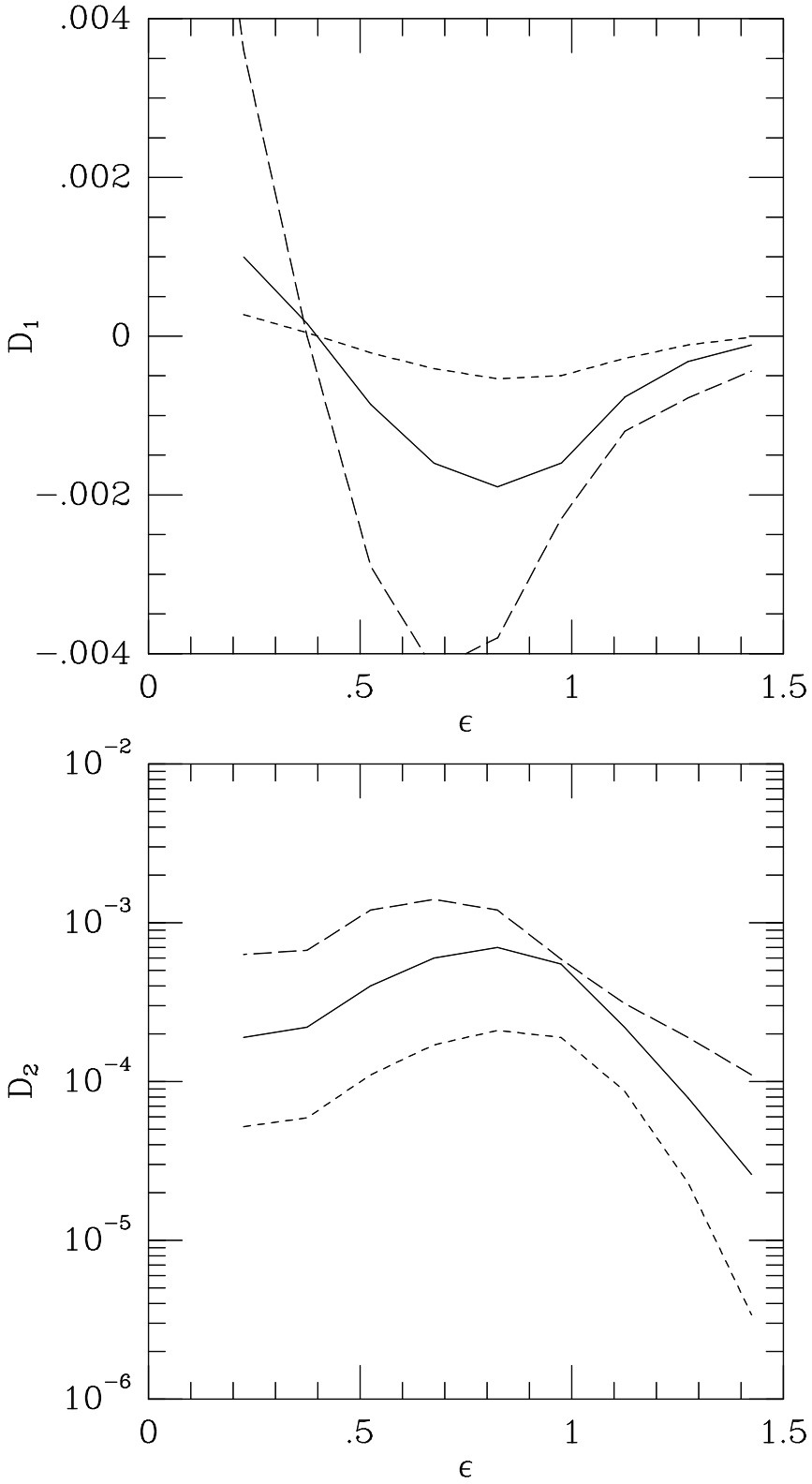
$$D_1 = \frac{< \varepsilon_i(t_0) - \varepsilon_i(t_0 + \Delta t) >}{\Delta t},$$

$$D_2 = \frac{< [\varepsilon_i(t_0) - \varepsilon_i(t_0 + \Delta t)]^2 >}{\Delta t}.$$

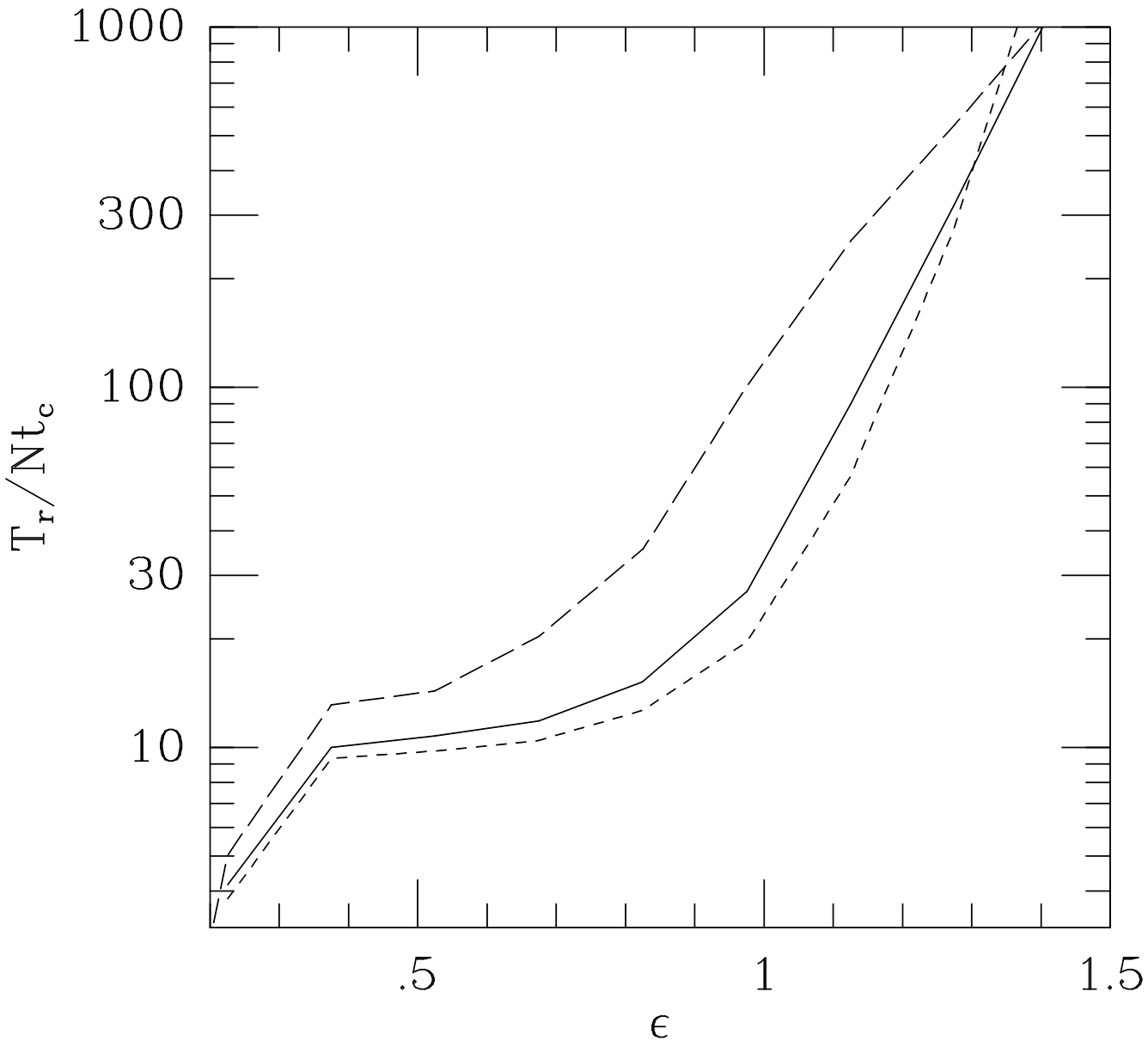
The inverse of them give the relaxation timescale. I use the definition

$$t_r = \varepsilon^2 / D_2,$$

Diffusion coefficients (continued)



Relaxation time



- **Relaxation is exponentially slow at high energy**
- **relaxation time $\propto N$**

Interpretation of "Chaotic itinerancy"

Relaxation in high-energy region is very slow

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Sheets with high energy perform random walk in very long timescale

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To obtain good statistic one has to wait very long

In other words....

There is nothing strange.

Why the relaxation in high energy is slow?

In three dimensions, particles with high energy have long relaxation time simply because the local density is low.

In one dimensions, density difference is rather small.

No clear explanation for the slow relaxation

In fact, it is not understood why sheet model relaxes at all.

— Random fluctuation of the field?

— Resonance?

Summary

- **The claimed strange behavior of the sheet model is an illusion. There is no distinction between microscopic and macroscopic relaxations.**
- **The relaxation timescale of mass sheets is really $O(N)$**
- **The relaxation time of the sheets in high-energy tail of the distribution function is surprisingly long, and we do not know the reason why.**